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# Generalising principles in spite of procedural differences: Children's understanding of division☆

Ekaterina Kornilaki[a], Terezinha Nunes[a,b,*]

[a] *University of Crete, Greece*
[b] *Department of Psychology, Oxford Brookes University, Gipsy Lane, Oxford OX3 0BP, UK*

**Abstract**

Between ages 5 and 7, children are known to be quite good at sharing discrete quantities but very bad at sharing continuous quantities. Our aim was to find whether they can transfer their understanding of logical relations from discrete to continuous quantities though the procedures used in sharing these quantities are markedly different.

Two samples of 5- to 7-year-olds participated in two studies. In the first study, the items involved partitive division; in the second, quotitive division tasks. In both studies, the children solved tasks with discrete and continuous quantities.

Performance varied significantly across age level and logical principle (equivalence between different rounds of sharing versus inverse relation between the divisor and the quotient) but not across type of quantity (discrete versus continuous). There was a very strong relation between performance across type of quantity. We conclude that children can generalise reasoning principles in division across type of quantity in spite of the difference in sharing procedures.


*Keywords:* Division procedures; Sharing continuous quantities; Sharing discrete quantities; Logic of division

The aim of the studies reported here was to analyse the connection between doing and understanding—more specifically, the connection between the ability to share things fairly and the understanding of logical principles related to division.

☆ This research was conducted as part of the first author's doctoral research.

* Corresponding author. Present address: Department of Educational Studies, University of Oxford. Fax: +44 1865 483887.

*E-mail address:* tnunes@brookes.ac.uk (T. Nunes).

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There is much evidence suggesting that 4- and 5-year-olds are able to share discrete quantities effectively and form equal size quotas (Davis & Hunting, 1990; Davis & Pepper, 1992; Davis & Pitkethly, 1990; Frydman, 1990). By the age of five, children can adjust their sharing procedures when they share different sized units (Frydman & Bryant, 1988). There is almost a consensus among researchers that the origin of children's understanding of partitive division is to be found in this early schema of sharing (Anghileri, 1997; Dickson, Brown, & Gibson, 1984; English & Halford, 1995; Fischbein, Deri, Nello, & Marino, 1985; Greer, 1992). The connection between sharing and division is a reasonable one because in both the action of sharing and the operation of division a quantity is divided into equal-sized quotas.

Correa, Nunes, and Bryant (1998) pointed out that, although the understanding of division originates from the action schema of sharing, division as an operation and the sharing schema are not the same thing. In a sharing situation the child can simply focus on procedures that will generate equal amounts for each recipient. A child who can do this will be credited with an understanding of sharing. In contrast, in order to attribute to children a basic understanding of the operation of division, one would expect them to grasp the relations between the three terms in a division situation: the dividend, the divisor and the quotient. That means that to credit children with an understanding of the basic ideas in division we would expect them to understand at least that there is a direct relation between the dividend and the quotient, when the divisor is kept constant, and an inverse relation between the divisor and the quotient, when the dividend is kept constant.

Consistently with this analysis, Correa et al. (1998) examined whether efficient sharing could be seen as a guarantee that children understood the inverse and direct relations between the three terms in division. They devised a set of experiments to test whether children who are able to share discrete quantities and infer the equivalence of the cardinal value of the shared sets can also understand the inverse divisor–quotient relation. They observed that many children who succeeded in sharing fairly and making numerical inferences regarding the equivalence of the shared sets failed to understand two principles in the relations between terms in division. The first one was an equivalence principle: equal dividends shared among equal number of recipients produce equal quotas. Only about two thirds of the 5-year-olds realised that if one shares the same amount to the same number of recipients in two separate rounds of sharing, the recipients in the second round will get just as much as those in the first round. The fact that one-third of the children failed to understand this equivalence principle is surprising because the participants had been screened for ability to share and to make inferences about the equivalence of the shared sets in a single round of sharing. The second principle was the inverse relation between divisor and quotient. Only about half of the 5-year-olds realised that the larger the number of recipients, the smaller their share. If instead of fixing the number of recipients first we actually fix the size of the share and ask the children what is the effect of different share sizes on the possible number of recipients, the level of difficulty increases: only 15% of the 5-year-olds performed above chance in this type of task. This is the negative side of these findings: knowing how to share is not sufficient for understanding division.

These findings can also be seen from a positive light: although young children are not taught about division in school, more than half of the 6-year-olds and the majority of the 7-year-olds realised that the more recipients in a sharing situation, the less they get and about half of the children in these age levels realised that the greater the share you give away, the fewer recipients will be able to get a share. It is thus possible that *knowledge of sharing procedures leads to an understanding of the relations between the three terms in a division situation*.

Correa (1994) further investigated the connection between children's implementation of a sharing procedure and their understanding of the inverse relation between the divisor and the quotient in a training experiment. She worked with two groups of children, in the age range 5–7 years. Both groups were asked about the inverse relation between the divisor and the quotient. The control group in her study simply answered questions about the relation between the size of the share and the number of recipients. The experimental group was asked to carry out the sharing for one of the groups of recipients before being asked to make the comparison across groups. The experimental group performed significantly better than the control group at all age levels and for both types of question regarding the divisor–quotient inverse relation. Thus, she concluded that the use of the sharing procedure to solve a practical task can help young children make progress in a non-computational, logical task about the inverse relation between the divisor and the quotient. Sophian, Garyantes, and Chang (1997) later replicated these results using a modified version of the task and of the sharing procedure. They asked the children to do the sharing after an answer had already been given so that sharing worked as a feedback for the answer. In their data analysis, they compared the children's performance in the later trials in the task with their own performance in the earlier trials. The children showed consistent and significant improvement in the later trials after having carried out the sharing in the earlier trials.

These findings can be interpreted as supporting a 'procedures first' hypothesis in the explanation of children's understanding of logical principles. According to this hypothesis, children learn to use procedures to solve problems and, through this experience, reflect about the relevant logical relations in the problem situation (see Rittle-Johnson & Siegler, 1998, for a review of research on the relation between procedural and conceptual knowledge).

The analysis of the 'procedures first' hypothesis in the context of sharing and division can offer a new contribution to the understanding of how procedural and conceptual knowledge are related because young children are quite successful in sharing discrete quantities but are very bad at sharing continuous quantities. Thus, it is important to know whether children can use their knowledge of principles in division, acquired with discrete quantities, when they are asked questions about continuous quantities. Can they transfer the principles learned with discrete quantities to continuous quantities though they are unable to share appropriately—i.e., lack the procedures for—the latter type of quantity?

A number of studies that investigated children's ability to share continuous quantities have documented their difficulties. Piaget, Inhelder, and Szeminska (1960) asked children to share circular and rectangular 'cakes' (represented by pieces of paper) among two, three, four and five dolls on different trials. They observed that many of the children could approximate a division of a rectangular piece of paper in half after some attempts. A major difficulty in the division into three parts was that many of children believed that they needed to cut across the rectangle three times so that each doll would receive one piece. This, of course, results in an extra piece, which the children either divided further and then shared out, or which was simply left over, resulting in a non-exhaustive division. Similar difficulties were observed with sharing the cake between five dolls.

So, in contrast to sharing discrete quantities where a procedure one-for-you one-for-me suffices, when sharing continuous quantities children must anticipate what they will do before the procedure is implemented. However, they seem to lack an anticipatory schema and checking procedures that are important for the effective sharing of continuous quantities (Hierbert & Tonnessen, 1978; Hunting & Sharpley, 1988a, 1988b; Miller, 1984). More recently, Empson (1999) confirmed the difficulties that 6-year-olds have with partitioning continuous quantities when the partitioning cannot be based on halving or repeated halving. Of the 15 children she interviewed, only 4 could indicate how they would share 2 candy bars among 3 children. The successful children indicated that they would divide each bar into three pieces and give one piece from each candy bar to each of the recipients. Though they did not necessarily succeed in executing this procedure nor did they use the correct fractional expression (thirds), they should be certainly credited with the use of an anticipatory scheme that allowed them to say how the sharing could be achieved. The remaining 9 children (60%) "partitioned the candy bars into halves or fourths with some leftover, 1 child used a strategy that consisted of creating random pieces represented by tallies, and 3 children did not solve this problem" (Empson, 1999, p. 298). Thus, Empson confirmed Piaget et al.'s (1960) earlier finding that the procedure for sharing continuous quantities cannot be done by simple rounds of sharing, as with discrete quantities: children must coordinate in an anticipatory fashion the number of sharers with the number of partitions.

This difference between the ease of procedures for sharing discrete quantities and the difficulty of procedures for sharing continuous quantities creates the opportunity for a new analysis of the 'procedures first' hypothesis. The evidence suggests that procedures for sharing discrete quantities support the development of children's understanding of logical relations in the division situation. If children between the ages of 5 and 7 years understand logical relations in division when the dividend is a discrete quantity, will they use this same logic to solve similar problems when the quantity to be divided is continuous, in spite of their notorious difficulty with sharing continuous quantities?

To answer this question we designed a study in which the children were asked to compare the quotients in different sharing situations involving either discrete or continuous quantities. If the children need to be able to implement sharing procedures that are specific to the type of quantity—that is, if they need to be able to share continuous quantities in order to reason about the division of continuous quantities—then we should find a significant difference between their performance in the division tasks as a function of type of quantity. However, if they can generalise their understanding of division from discrete to continuous quantities, there should be no difference between their performance as a function of quantity type. Instead, there should be a strong association between performance in the discrete and continuous quantities tasks.

Children's understanding was examined in two types of division problem, partitive and quotitive division. In partitive division problems, the divisor is the number of recipients and the quotient is the share they receive. In quotitive division problems, the divisor is the share to be given to each recipient and the quotient is the number of recipients. The problems we posed to the children, like those used by Correa et al., did not require arithmetic computations, but simply an understanding of the relation between the size of the divisor and the size of the quotient.

## 1. Experiment 1

### *1.1. Method*

#### *1.1.1. Participants*

The participants were: (a) 32 five-year-olds (16 boys and 16 girls; mean age 5 years and 5 months; age range: 5 years to 5 years and 11 months); (b) 32 six-year-olds (14 boys and 18 girls; mean age 6 years and 6 months; age range: 6 years to 6 years and 10 months); and (c) 32 seven-year-olds (15 boys and 17 girls; mean age 7 years and 6 months; age range: 7 years to 7 years and 11 months). The children attended two state-supported schools in Northeast London. According to the information given by the class teachers, they had not received instruction on division in school.

### *1.2. Experimental tasks*

The children were asked to make judgements about the relative size of the shared quotas in problems that involved either discrete or continuous quantities. The recipients in all the problems were cats, represented by pictures of white and brown cats, organised in separate groups. The discrete quantities to be shared were sets of small fish. The continuous quantities to be shared were fishcakes. In each trial, the child was shown the number of small fish or the fishcakes to be shared. There was always a representation of the dividend to be shared among the white cats and a separate representation of the dividend that was to be shared among the brown cats. The dividend was equivalent for the two groups of cats in every trial. The child's task was to say whether the white cats would receive the same size share as the brown cats and, if not, which cats would receive a larger share.

For each type of quantity, the number of recipients was systematically varied in order to produce two conditions. (1) In the *same condition* the size of the divisor was the same, i.e., there were as many white as brown cats. For example, the children were shown two groups of 3 cats each; each group would be sharing 12 fish. We use the notation 'trial 12 3(3)' to refer to this trial. (2) In the *different condition* the number of recipients varied. For example, in one group there were 3 cats sharing 12 fish and in the other group there were 2 cats sharing 12 fish (trial 12 3(2)). Equivalent problems of both types were presented with continuous quantities, when the dividend was composed of fishcakes.

If the children understood the inverse relation between the divisor and the quotient, they could reason that, the fewer the cats, the more they will receive independently of whether the dividend is a set of small fish or fishcakes.

The size of the shared quantity varied across trials. The number of fish to be shared in the discrete quantities trials was either 12 or 24. These numbers were chosen because each of them has allowed for the generation of at least four divisor–quotient pairs and it was possible to check eight times the consistency of children's responses. With continuous quantities the size of the dividend varied and the cats might be sharing 1 (unitary fraction), 2 or 3 (non-unitary fractions) fishcakes; the number of fishcakes was always smaller than the number of recipients, resulting in the need to consider how continuous quantities might be shared. Because there is evidence suggesting that children have a better understanding of unitary than of non-unitary fractions (Goldblatt & Raymond, 1996), both types of situation were included in the task. If the children could order the size of the quotas on the basis of the relations between the division terms, the size of the dividend was not expected to affect the difficulty of the task. In the different-divisor condition we also considered the effect of the numerical contrast between the alternatives. There were trials that had a small (2 versus 3 cats) and pairs with a bigger (3 versus 6 cats) difference between the divisors. The trials where the numerical difference between the divisors was greater were expected to encourage children's thinking about the inverse divisor–quotient relation and thus might prove easier.

There were 16 trials in the discrete quantities task, four trials with each dividend, in each of the conditions, same or different divisors. The number of cats in the groups of recipients varied from 2 to 6, generating 8 trials in each condition. There were 24 trials in the continuous quantities tasks, 4 trials with each dividend (1, 2 or 3 fishcakes) in each of the conditions, same or different divisors. The number of cats sharing in the problems with unitary fractions varied between 2 and 6 and in the non-unitary condition it varied between 4 and 9. Table 1 presents a summary description of the trials.

Because our aim was to assess the children's logical understanding of the relation between the divisor and the quotient, the materials were used only to present the problems. The children did not manipulate the material to find out the result.

### *1.2.1. Materials*

The materials consisted of 18 paper-cut identical cats (9 brown and 9 white) of the same size, 48 paper-cut identical bluish-grey fish and 6 identical paper-cut round cakes.

Table 1
Trials used in Experiment 1

| Same-condition trials | Different-condition trials |
|---|---|
| Discrete quantities | |
| 12 2(2) | 12 2(3) |
| 12 3(3) | 12 2(6) |
| 12 4(4) | 12 3(4) |
| 12 6(6) | 12 3(6) |
| 24 2(2) | 24 2(3) |
| 24 3(3) | 24 2(6) |
| 24 4(4) | 24 3(4) |
| 24 6(6) | 24 6(3) |
| Continuous quantities | |
| 1 2(2) | 1 2(3) |
| 1 3(3) | 1 2(6) |
| 1 4(4) | 1 3(4) |
| 1 6(6) | 1 3(6) |
| 2 4(4) | 2 4(6) |
| 2 5(5) | 2 4(5) |
| 2 6(6) | 2 5(7) |
| 2 7(7) | 2 6(7) |
| 3 6(6) | 3 6(8) |
| 3 7(7) | 3 6(7) |
| 3 8(8) | 3 9(7) |
| 3 9(9) | 3 9(8) |

### 1.2.2. Procedure

The children were seen individually in their school premises in two sessions held on the same day. In one session they were asked about discrete quantities and in the other about continuous quantities. The order of sessions was systematically varied, with half the children participating in the discrete quantities task first and the other half in the continuous quantities task first. In the discrete quantities task, the children were presented with two groups of cats, the brown and the white cats. They were told that the cats were going to eat their favourite dinner, fish. There were, for example, 12 fish for each group. The experimenter counted the number of fish in each set out loud to ensure the children of the equality, and placed each set in a pile to avoid responses based on correspondence procedures. Then the children were told that the white cats were going to share their fish fairly and eat all the fish up and that the brown cats were going to do the same, share their fish fairly and eat all the fish up. Then the experimenter pointed to one white and one brown cat and asked whether they would receive the same or a different amount of fish. If the answer was 'different', they were asked to indicate which one would receive more. The children were also asked to justify their responses independently of whether they were correct or not. The same procedure was followed with continuous quantities, but this time the quantity to be shared was one or more fishcakes.

## 1.3. Results

One point was given to each correct response. Total scores were obtained for each condition (same or different number of divisors) and type of quantity (discrete or continuous). The distribution of scores in all the cells was highly skewed: the children either passed or failed all the items. In the same-divisors condition, both for discrete and for continuous quantities, the distributions were negatively skewed, with the majority of the children passing all the items: 62% of the 5-year-olds, 84% of the 6-year-olds and all the 7-year-olds answered all the questions correctly. In the different-divisors condition, the distribution was bimodal. Thus, instead of using scores for subsequent analyses, the children were given a pass score if they performed above chance and a failing score if they did not.

The probability of correct responses occurring by chance was calculated for each condition. In each trial the child could provide one of three possible responses: (a) the cats will receive the same amount; (b) the white cats will receive more; or (c) the white cats will receive less. There were eight trials in each condition with discrete quantities and three possible choices in each trial. Above chance scores were 6 or more correct answer out of the 8 trials ($p < .01$). There were 12 trials in each condition with continuous quantities; above chance scores were 9 or more out of 12 for the continuous quantities ($p < .01$).

The number of children who performed significantly above chance in the same- and the different-divisor conditions with discrete and continuous quantities is shown in Table 2.

Children's success in the same-divisor condition suggests that the majority of the 5-year-olds could use the equivalence principle. The rate of success for 5-year-olds in this condition was approximately 2/3 both for continuous and discrete quantities. Thus, we replicate Correa et al.'s results for discrete quantities and extend their finding to the use of this principle with continuous quantities. A chi-square test showed that there was a significant positive association between age and performance in the discrete quantities task ($\chi^2 = 15.58$,

Table 2
Number of children who scored above chance in each condition by age and type of quantity

| Age level | $n$ | Type of quantity | Condition | |
|---|---|---|---|---|
| | | | Same divisor | Different divisor |
| 5 | 32 | Discrete | 20 | 11 |
| 6 | 32 | | 27 | 17 |
| 7 | 32 | | 32 | 26 |
| 5 | 32 | Continuous | 20 | 10 |
| 6 | 32 | | 27 | 16 |
| 7 | 32 | | 32 | 26 |

d.f. = 2, $p < .0001$) and also between age and performance in the continuous quantities task ($\chi^2 = 15.58$, d.f. = 2, $p < .0001$). These findings are indicative of a developmental trend though the majority of the children already used the equivalence principle at age five.

Children's success in the different-divisors conditions also shows improvement with age for discrete ($\chi^2 = 14.48$, d.f. = 2, $p < .001$) and continuous quantities ($\chi^2 = 16.45$, d.f. = 2, $p < .0001$).

The different-divisors condition was clearly more difficult than the same-divisor condition. Table 3 shows the number of children who succeeded in both tasks, failed both tasks, or passed the same-divisor but failed the different-divisors condition. There were no children who passed the different-divisors condition but failed the same-divisor condition. Thus, the inverse relation between divisor and quotient is understood later than the equivalence principle in division.

The level of success in both conditions was very similar for the two types of quantities. In order to investigate whether the children's performance varied as a function of the type of quantity, we analysed the number of children who passed both tasks, failed both tasks, or passed one task but not the other. Table 4 shows the number of children who succeeded and failed the inverse relation tasks as a function of type of quantity.

A McNemar test showed that the type of the shared quantity did not affect children's performance ($p < .5$). The performance in the two tasks was highly correlated: the children who were successful with discrete and continuous quantities were the same, apart from two, who were successful with discrete quantities only. Thus, we conclude that children's knowledge of the sharing principles for discrete quantities allows them to reflect about the logical properties of division irrespective of the type of quantity in the division situation. This

Table 3
Children's performance across conditions by type of quantity

| Age level | $n$ | Type of quantity | Fail both conditions | Pass same fail different | Pass both conditions |
|---|---|---|---|---|---|
| 5 | 32 | Discrete | 12 | 9 | 11 |
| 6 | 32 | | 5 | 10 | 17 |
| 7 | 32 | | – | 6 | 26 |
| 5 | 32 | Continuous | 12 | 10 | 10 |
| 6 | 32 | | 5 | 11 | 16 |
| 7 | 32 | | – | 6 | 26 |

Table 4
Number of children succeeding and failing in the different-divisors condition by type of quantity

| | Discrete quantities | | Total |
|---|---|---|---|
| | Pass | Fail | |
| Continuous quantities | | | |
| Pass | 52 | – | 52 |
| Fail | 2 | 42 | 44 |
| Total | 54 | 42 | 96 |

conclusion is justified in view of children's well-documented failure to share continuous quantities at this age level.

Finally, in order to provide qualitative evidence regarding the children's reasoning, their justifications in the different-divisors condition were categorised into five descriptive categories derived from the observed justifications. Two independent judges categorised the responses; the inter-judge agreement was 98%.

The types of justification were as follows.

I: *No justification* ("I don't know"), *irrelevant or idiosyncratic responses* ("the white cat is more hungry").

II: *Same dividend, same share*: Children who offered this type of justification focused their attention on the size of the dividend, without taking into account the number of recipients ("they will eat the same; both have 12 fish").

III: *The more cats, the greater the share*: In this case the children established an incorrect direct relation between the number of recipients and the size of the quotas. They applied the "more-is-more" rule (a child who indicated that the cat sharing with a larger number of friends would distribute larger shares explains that this is so "because there are more cats here"; see also Stavy & Tirosh, 2000).

IV: *Quantification*: Very few children justified their responses by attempting, not always successfully, to quantify the quotient and offered a number as a justification ("this cat will get 3, but this one will get 2).

V: *Inverse relation reasoning*: Children who offered this type of response found it sufficiently clear to refer to the inverse divisor–quotient relation: the fewer the cats sharing, the more they will get ("this one will get more, because here are less cats").

Children's justifications in the different-divisors condition were analysed across the age groups with discrete and continuous quantities. Table 5 shows the proportion of justifications by type of quantity. About one-third of the 5-year-olds gave either no justification or an irrelevant justification (type I). This absence of justification decreased markedly with age. About one-third of the 5- and 6-year-olds applied the more-is-more rule (type III); there was a sharp decline in the use of this type justification by the age of 7. There were very few attempts to quantify the division (type IV). There is a consistent increase with age in the proportion of justifications based on the inverse relation between the divisor and the quotient (type V).

The proportion of children who justified their responses by referring to the inverse relation is very similar to the proportion of children giving correct answers in the different-divisor

Table 5
The proportion of justifications with discrete and continuous quantities in the different-divisor condition

| Age | n | Discrete quantities | | | | | Total |
|---|---|---|---|---|---|---|---|
| | | I | II | III | IV | V | |
| 5 | 32 | .36 | .04 | .29 | .01 | .30 | 1 |
| 6 | 32 | .12 | .03 | .35 | .02 | .48 | 1 |
| 7 | 32 | .04 | .02 | .13 | .06 | .75 | 1 |
| **Age** | ***n*** | **Continuous quantities** | | | | | **Total** |
| | | I | II | III | IV | V | |
| 5 | 32 | .32 | .04 | .34 | .01 | .29 | 1 |
| 6 | 32 | .09 | .03 | .33 | .02 | .53 | 1 |
| 7 | 32 | .06 | .01 | .13 | .03 | .77 | 1 |

*Note*. I: no or irrelevant justification; II: focus on the equality of dividends; III: direct divisor–quotient relation; IV: attempt to quantify; V: inverse divisor–quotient relation.

tasks: about one-third of the 5-year-olds, half of the 6-year-olds and three quarter of the 7-year-olds gave correct answers and justified their choices on the basis of the inverse relation between divisor and quotient. This suggests that their choice responses in the task were based on an explicit understanding of the inverse principle. There are no appreciable differences between the justifications used for continuous and discrete quantities. Thus, the analysis of justifications provides further support to the hypothesis that, having understood the inverse relation in the context of discrete quantities, the children apply the same reasoning to continuous quantities, which they are so notably bad at sharing.

### *1.4. Discussion*

The aim of this experiment was to examine whether children who understand the logical relation between the terms of division in the context of discrete quantities can also apply the same reasoning with continuous quantities. The interest of this comparison stems from the fact that quite young children can share discrete quantities successfully but they are remarkably bad at sharing continuous quantities. The results clearly show that children's use of the equivalence principle and of the inverse relation between divisor and quotient in problem solving is independent of the type of quantity shared.

This finding is provocative because it raises questions about the very notion of procedures as the basis for understanding logical principles. The argument goes as follows. Previous work suggested that sharing—conceived as a *procedure* such as one-for-you, one-for-me, which ensures the equality of the quotas—facilitates reasoning about the equivalence and the inverse principles in division, supporting the 'procedures first' hypothesis. Our study shows that, when these principles are applied to discrete quantities, they can also be applied to continuous quantities. In fact, although there is an order in the acquisition of procedures for sharing, we could not detect a gap between the use of logical principles for discrete and continuous quantities. Thus, *there need not be a specific connection between the procedure* *and the quantities in division*. This lack of specificity calls into question the view that it is the procedure itself that matters.

Procedures are specific representations of ‘how to’ solve a problem. Sometimes these representations can be conceived in more abstract ways but sometimes they must be conceived in specific ways. For example, procedures for counting can be general: the one-to-one correspondence principle can be applied irrespective of the type of object. However, in the case of sharing, the procedure one-for-you, one-for-me cannot be applied to continuous quantities. Different procedures are required for sharing discrete and continuous quantities. Yet, understanding the equivalence and the inverse principles in division in the context of discrete quantities suffices to promote the same reasoning in the context of continuous quantities.

Piagetian theory offers a different conceptualisation of the relation between doing and knowing through the concept of schemas of action. The difference between procedures and schemas of action is that a schema of action does not represent what is specific in a situation: it only represents the relations that can be applied across situations.

Our results strongly suggest that children’s judgements of the size of the shared quotas are based on the understanding of logical relations between the division terms irrespective of the differences in procedures that would be used for sharing the different types of quantity. This conclusion is further strengthened by the fact that children’s performance was not affected by the size of the shared quantity. If children’s reasoning were dependent on their ability to quantify the division sums, then their performance would have varied both as a function of type of quantity and the size of the share, because they are more successful in sharing smaller than larger quantities. Further support for the idea that the children are reasoning in terms of logical relations comes from the analysis of their justifications: children’s justifications stressed the relation between the division terms, not the procedure used to obtain shares.

At about the age of 6 or 7, children seem to take a great step towards the understanding of the inverse divisor–quotient relation. Our results concerning children’s ability to infer the relative size of the shared quotas with discrete quantities are largely in accordance with Correa et al’s (1998) findings. Both studies showed that by the age of 7 the majority of children could anticipate what the results of sharing will be by applying the inverse divisor–quotient relation. This achievement is indeed impressive in view of the fact that children in English schools only learn about the operation of division almost a year later.

These results are also puzzling to some extent. Much research (e.g., Elkind, 1967; Piaget & Inhelder, 1974; Price-Williams, Gordon, & Ramirez, 1969) has documented that the type of quantity affects children’s performance in conservation tasks. Thus, it is puzzling that this is not the case with respect to the inverse relation between the divisor and the quotient. At present we have no explanation for this discrepancy in results across logical principles.

## 2. Experiment 2

The aim of this experiment was to assess whether the conclusions drawn in Experiment 1 with respect to children’s understanding of partitive division can be generalised to their understanding of quotitive division. Correa et al. (1985) have shown that children’s understanding of the equivalence and the inverse principles in quotitive division is achieved later than the same understanding in partitive division. Our study investigates whether children's performance in quotitive division tasks that involve the equivalence and the inverse principles differs across discrete and continuous quantities tasks.

### *2.1. Method*

#### *2.1.1. Participants*

The participants were (a) 32 five-year-olds (16 boys and 16 girls; mean age 5 years and 7 months; age range: 5 years to 5 years and 11 months), (b) 32 six-year-olds (15 boys and 17 girls; mean age 6 years and 7 months; age range: 6 years and 1 month to 6 years and 11 months), and (c) 32 seven-year-olds (16 boys and 16 girls; mean age 7 years and 6 months; age range: 7 years to 7 years and 10 months). The children attended two state-supported schools in the same educational zone as the children of the previous experiment. According to their teachers, they had not received formal school instruction in division.

#### *2.1.2. Experimental tasks and procedure*

The design in this experiment is analogous to that in Experiment 1 but this time the children were asked to make judgements regarding the relative number of recipients when the size of the shares was fixed. In the same-divisor condition the size of the dividend and the size of the shared quotas was the same. For example, the children were shown 2 piles of 12 fish. Each pile belonged to one cat, white or brown, who was going to share his fish to his friends. The white cat wanted to share the fish in lots of two and the brown cat wanted to share the fish in lots of two as well. The children were asked whether the white cat and the brown cat would be able to give these treats to the same number of friends. If the answer was a different number of friends, they would be asked which cat would be able to treat more friends. In the different-divisors condition the size of the dividend was the same for both cats and the size of shared quotas varied. For example, the white cat wanted to share the 12 fish in lots of three and the brown cat wanted to share his fish in lots of two. The children were again asked whether the white cat and the brown cat would be able to treat the same or a different number of friends. In order to aid the children's memory a drawing of a plate with the correct number of fish in each share was placed next to the cats when each trial was explained.

In the task with continuous quantities the cats were going to share pieces of fishcake from one of more fishcakes. The dividend was always the same. In the same-divisor condition the pieces of fish cake were of the same size; in the different-divisor condition they were of different sizes. One of the slices to be shared was placed by each cat to illustrate the size of the pieces. The question was whether the white cat and the brown cat would be able to treat the same number of friends to pieces of fishcake. If the children said ‘no’, they were asked which cat would be able to treat more friends.

In the discrete quantities tasks the size of the dividend varied between 12 and 24 and the size of quotas between 2 and 6. With continuous quantities the number of fishcakes varied between 1 and 3; the size of quotas was either half, one-third, one-quarter or one-eighth of a cake. The order of the tasks involving different types of quantities was systematically varied across children: half of the children answered the discrete quantities task first and the other half answered the continuous quantities task first.

The number of trials with discrete and continuous quantities was the same as in Experiment 1. The children were tested individually at school. After each response they were asked to justify their answer. Because we wanted logical rather than empirical solutions to the questions, the children did not have the option of sharing out the fish or cutting the cakes into the stipulated slice-sizes.

#### *2.1.3. Material*

The material consisted of 2 paper-cut cats that differed only in colour, one brown and one white, 48 paper-cut identical bluish-grey fish, 6 identical round paper-cut cakes, pictures of plates having either 2, 3, 4 or 6 fish, and slices of paper-cake equal to half, one-third, one-quarter and one-eighth of the round cakes.

### *2.2. Results*

For each correct response the child was granted one point. As in the previous experiment, the distributions were highly skewed and it was not possible to work with scores. We used the same criteria for categorizing children into those who scored above and those who scored below chance. As in the previous experiment the size of the dividend or the size of the divisors did not affect children's responses; responses are thus grouped by type of quantity and type of question. The number of children who performed above chance level is presented in Table 6. As can be seen, the different condition was harder. Not all the children who succeeded in the same condition performed equally well in the different condition, but the children who did well in the different condition always succeeded in the same condition.

Chi-square tests revealed a significant improvement in performance with age both in the same-quota trials with discrete and continuous quantities ($\chi^2 = 24.5$, d.f. = 2, $p < .0001$) and in the different-quotas trials with discrete ($\chi^2 = 18.92$, d.f. = 2, $p < .0001$) and continuous quantities ($\chi^2 = 16.20$, d.f. = 2, $p < .0001$).

Children's performance was also compared across the type of the quantity. Similarly to the findings in Study 1, all the children who were successful with discrete quantities did well with continuous quantities. However, there were five children who succeeded with continuous quantities, but failed with the discrete quantities task. A McNemar test showed that there was no significant difference in performance across the two types of quantities but the difference was close to significant ($p = .063$). This trend, however, does not suggest that

Table 6
Number of children who scored above chance in each condition by age and type of quantity

| Age | *n* | Type of quantity | Condition | |
|---|---|---|---|---|
| | | | Same divisor | Different-divisor |
| 5 | 32 | | 16 | 5 |
| 6 | 32 | Discrete | 27 | 12 |
| 7 | 32 | | 32 | 22 |
| 5 | 32 | | 16 | 7 |
| 6 | 32 | Continuous | 27 | 14 |
| 7 | 32 | | 32 | 23 |

Table 7
Proportion of justifications with discrete and continuous quantities in the different-dividends condition

| Age | *n* | Discrete quantities | | | | | Total |
|---|---|---|---|---|---|---|---|
| | | I | II | III | IV | V | |
| 5 | 32 | .27 | .03 | .55 | – | .15 | 1 |
| 6 | 32 | .09 | – | .53 | .02 | .36 | 1 |
| 7 | 32 | .03 | – | .28 | .06 | .63 | 1 |
| **Age** | ***n*** | **Continuous quantities** | | | | | **Total** |
| | | I | II | III | IV | V | |
| 5 | 32 | .29 | .03 | .48 | .02 | .18 | 1 |
| 6 | 32 | .06 | – | .49 | .03 | .42 | 1 |
| 7 | 32 | .02 | – | .25 | .07 | .66 | 1 |

*Note.* I: no or irrelevant justification; II: focus on the equality of dividends; III: direct divisor–quotient relation; IV: attempt to quantify; V: inverse divisor–quotient relation.

the children failed to transfer the principles understood in the context of sharing discrete quantities to continuous quantities: the performance was slightly better with continuous rather than discrete quantities. Therefore, it is the association between the performance in the tasks with different quantities, rather than the difference between the two, that is the significant result.

### *2.2.1. Children's justifications*

In order to get a better insight into children's reasoning, their justifications were analysed. The same categories developed for classifying the responses in the previous experiment were used in this experiment. The inter-judge agreement between two independent judges was 96%.

Table 7 shows that about half of the 5- and 6-year-olds focused on the divisor–quotient relation but used a direct proportional rule: the more you give away, the more friends you can treat (type III). This type of justification decreased in frequency at the age of 7. The number of irrelevant responses also decreased with age (type I). There were very few attempts of quantification (type IV) and few children focused on the equality of the dividends in the sharing situations (type II). The correct justification based on the inverse relation between the dividend and the quotient (type V) was the most frequent at age 7: about two thirds of the 7-year-olds used this type of justification.

A further analysis showed a strong association between performance and type of justification both for discrete and continuous quantities (Table 8). Correct responses were accompanied by correct reasoning and wrong responses by inappropriate justifications.

## 3. General discussion

Our studies replicate previous results and also provide new evidence regarding the ability of young children to understand the inverse relation between the divisor and the quotient

Table 8
Proportion of justifications of different types as a function of performance and by type of quantity

| Performance | Type of quantity | | | | | Total |
|---|---|---|---|---|---|---|
| | Discrete quantities | | | | | |
| | I | II | III | IV | V | |
| Fail | .20 | .02 | .78 | – | – | 1 |
| Succeed | .01 | – | – | .05 | .94 | 1 |
| Performance | Continuous quantities | | | | | Total |
| | I | II | III | IV | V | |
| Fail | .21 | .02 | .77 | – | – | 1 |
| Succeed | .04 | – | – | .04 | .92 | 1 |

*Note*. I: no or irrelevant justification; II: focus on the equality of dividends; III: direct divisor–quotient relation; IV: attempt to quantify; V: inverse divisor–quotient relation

with continuous quantities. Four aspects of our results and their implications are highlighted here.

First, we were able to replicate Correa et al's (1998) findings on young children's ability to reflect on the relation between the division terms in discrete quantities situations. Both studies have presented convincing evidence that, at the age of 6 and 7, children have a good insight into the relations between the division terms, long before they are introduced to this operation at school. This result supports the view that children learn a considerable amount about mathematical reasoning outside school (see, for example, Carpenter & Moser, 1982; Ginsburg, Klein, & Starkey, 1998; Saxe, Guberman, & Gearhart, 1987) and that this informal knowledge is not restricted to principles involved in addition and subtraction.

Secondly, we extended Correa et al.'s (1998) findings by documenting children's ability to reflect on the relation between the division terms in the context of continuous quantities. The effect of problem type, partitive versus quotitive, on the understanding of the inverse relation between divisor and quotient was replicated with continuous quantities: children find it easier to reason about the inverse relations in partitive than in quotitive situations. However, within each type of problem, there was no difference in the children's performance as a function of type of quantity: they were as successful in discrete as in continuous quantities problems.

Thirdly, the fact that children's performance was not affected by the type of the quantity suggests that understanding the consequences of sharing something between different numbers of recipients is not dependent on a specific procedure for sharing. Continuous quantities cannot be shared using the procedure one-for-you one-for-me that children use to share discrete quantities. Thus, the connection between doing and knowing—procedures and understanding—may not always follow a single direction. Children are able to share discrete quantities and seem to derive an understanding of the inverse divisor–quotient relation from it. They can then transfer this understanding across to continuous quantities at an age when they are notoriously bad a sharing continuous quantities. It is quite possible that this understanding will then play a role in the ability to anticipate the results of sharing before they can actually succeed in sharing continuous quantities. However, our study does not provide evidence for this latter hypothesis, and further research is necessary to examine it.

The children's success in understanding the inverse relation between the quotient and the divisor should not be seen as a trivial achievement. It has been argued that children develop their first mathematical understandings from intuitions that involve extrapolations beyond facts (Fischbein, 1987). These intuitions are said to result often in misconceptions. One such misconception is the use of intuitive rules such as "the more A, the more B" or "the same A, the same B" (e.g., Stavy & Tirosh, 2000). Behaviours consistent with these rules could and were observed in our study when the children were solving the problems involving discrete and also problems involving continuous quantities. Very few justifications were consistent with the "same A, same B" rule, which would be the one used by the children who focused on the fact that the dividend was the same in the sharing situations being compared: the rate of children giving this type of justification never reached 5%. Justifications consistent with the rule "the more A, the more B" appeared significantly more often, and the percentage of children offering this justification varied with the level of task difficulty: it was more frequent in the quotitive division tasks than in the partitive division tasks. In the partitive division tasks, one-third of the 5- and 6-year-olds justified their responses as "the more recipients, the more they get", but this response decreased markedly with age as only slightly more 10% of the 7-year-olds used this incorrect direct proportional reasoning. Significantly more children gave this type of response in the quotitive reasoning tasks: about half of the 5- and 6-year-olds and one-quarter of the 7-year-olds answered that the more the cat gave to each friend, the more friends he could have at his party. This was actually the justification used by three quarters of the children who gave wrong answers. Almost all of the children who gave correct responses justified their answer on the basis of an inverse relation between the divisor and the quotient.

It could be argued that these results provide some support for the intuitive rules theory. However, we consider our results with caution in this respect. The frequency of use of these intuitive rules was affected by the level of difficulty of the task. Thus, children who used the incorrect direct proportional reasoning in the quotitive division task had not used it in the partitive division task. Thus, it would be difficult to argue that the children in our study simply *reasoned by following these rules*. Here we echo a point made by Van Dooren, Weyers, and Vershaffel (2004) in an analysis of more difficult tasks with secondary school pupils: if the type of task influences the pupils' responses, it is unlikely that they are blindly using these rules. It is more likely that pupils attempt to understand the problems, and that this incorrect analysis is an outcome of their attempt, rather than an unjustified extrapolation from other situations where this direct relation would apply. Piaget (1970; first published in 1946) documented similar examples of the use of incorrect direct proportional reasoning in his studies of children's understanding of movement and speed. He observed that a child who correctly understands that "the greater the distance, the more time you need to cover it" and that "the greater the speed, the more distance you can cover", might wrongly conclude that "the greater the speed, the greater the distance you can cover, and because you can cover a greater distance, you need more time". He described this form of reasoning as "transductive reasoning", giving the child recognition for the ability to reason in a step-by-step fashion but noting the child's failure in building a consistent system of relations between the different variables that come into play in the situation. He contrasts transductive reasoning with deductive reasoning, in which the child is able to integrate all the elements into a single analysis, and thus realise that, *if the distance is kept constant*, the greater the speed, the less time you need. It is quite possible that the children in our study also reasoned in this step-by-step way: the more friends the cat invites, the more food he needs; the more food he has, the larger the portions; the more friends he invites, the more food he has, so the larger the portions. However, our study was not designed to test these alternative theories so we cannot make further progress in the assessment of the intuitive rules versus the transductive reasoning explanation. This is a topic well worth pursuing in further research.

Finally, we wish to point out an important educational implication of our results. Most of the 7-year-olds in our study understood that in sharing situations there is an inverse relation between the divisor and the quotient, irrespective of whether the quantities shared are discrete or continuous quantities. In the continuous quantities situations we investigated, the problems can be represented as comparisons, for example, between 1/3 and 1/4, when the white cat was sharing one fish-cake for 3 cats and the brown cat was sharing one fish cake for 4 cats. This insight into the informal, contextualised problem, contrasts with the notorious difficulty of older pupils in ordering fractions: a very common misconception reported in the literature (Kerslake, 1986; Mack, 1995) is to say that 1/4 is more than 1/3 because 4 is more than 3. This frequent error occurs when children are asked to compare the numerical fractional representations, without establishing a connection between sharing and fractions. Mack (1990) has actually shown that *the same children* can say that when one pizza is shared among 3 children and one identical pizza is shared among 4 children, the children in the first group receive more—and then go on to say immediately afterwards that 1/3 is less than 1/4. The lack of connection between children's informal knowledge of this aspect of the logic of fractions and their school knowledge of fractions is reminiscent of other studies that have documented the same phenomenon in the domain of whole numbers (e.g., Nunes, Schliemann, & Carraher, 1993). Thus, comprehensive assessments of children's understanding of fractions should include problems that could be solved by children using their informal knowledge as well as the more traditional school tasks of numerical comparisons and calculation procedures.

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